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HAMILTONIAN SYSTEMS OF CALOGERO TYPE AND TWO DIMENSIONAL YANG-MILLS THEORY

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Abstract

We obtain integral representations for the wave functions of Calogero-type systems, corresponding to the finite-dimensional Lie algebras, using exact evaluation of path integral. We generalize these systems to the case of the Kac-Moody algebras and observe the connection of them with the two dimensional Yang-Mills theory. We point out that Calogero-Moser model and the models of Calogero type like Sutherland one can be obtained either classically by some reduction from

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two dimensional Yang-Mills theory with appropriate sources or even at quantum level by taking some scaling limit. We investigate large k limit and observe a relation with Generalized Kontsevich Model.

1 Introduction

It is well known that in some supersymmetric quantum mechanical theories it is possible to evaluate path integrals exactly. Usually it is the case when bosonic part of the action of this theory can be interpreted as a Hamiltonian of circle action on the loop space, provided that the symplectic structure on the loop space is defined by the fermionic part of the action. Then algebra of the supersymmetry is interpreted as an equivariant derivative action and path integral defines an equivariant cohomology class which in the nicest situation localizes on the fixed points of circle action and gives finite-dimensional integral as an answer. All this can be generalized to the case of non-abelian group action [1],[2].

It is a general belief that the localization technique can be applied to the analysis of integrable systems. Namely it is expected that partition functions and correlators in systems with finite or even infinite number degrees of freedom can be calculated in this manner [3]. So it is natural to start from the quantum mechanical systems related with the finite dimensional Lie groups. There are two large classes of such systems, namely generalized Toda chains or lattices and generalized Calogero-Moser models. Both of them possess a very transparent algebraic phase space and can be described as a different projections of the free motion on the curved manifold. These systems in some sense are the particular examples of the multimatrix models but the underlying Hamiltonian structure of these models provides one with a possibility to handle with them in a canonical manner.

The multiparticle quantum mechanical systems evidently should be generalized to the field theories. This program has been done for the Toda

system which smoothly transforms from quantum mechanics to the Toda field theory. The underlying structure is the Kac-Moody algebra, since the Poisson brackets in the field theory correspond to the symplectic structure on the Kac-Moody coadjoint orbit. We shall show that there exists also the field theory generalization of Calogero-Moser systems. It appears that these are nothing but two-dimensional Yang-Mills theory with the appropriate sources.

The paper is organized as follows. In section 2 we consider as an example Calogero-Moser dynamical system and evaluate exactly the corresponding path integral. Then we generalize this result to the case of generic root system. In section 3 we turn to the infinite-dimensional situation and investigate two-dimensional Yang-Mills system and its generalization in the spirit of the previous constructions. In section 4 we briefly discuss supersymmetric part of this stuff. In section 5 we discuss some open problems, concerning relations with the rational solutions of KP hierarchies. In section 6 we present our conclusions.

2 Calogero-Moser system

2.1 Classical Calogero-Moser system as a hamiltonian reduction of the free system

Classical Calogero-Moser system is a system of N particles on the real line with the pair-wise interaction potential [4]:

$$V(x_1, \dots, x_N) = \sum_{i \neq j} \frac{g^2}{(x_i - x_j)^2}, \quad (2.1)$$

g^2 is a coupling constant, which is supposed to satisfy $g^2 > -1/8$, to avoid the collapse of the system. It is well known [5],[6] that such a system (and all generalizations, like Sutherland's one with $\frac{1}{\sin^2(x_i - x_j)}$ instead of $\frac{1}{(x_i - x_j)^2}$ or with extra quadratic potential) appears as a result of hamiltonian reduction of some simple hamiltonian system. For pure Calogero system (2.1) it goes

as follows. We start from the free system on the cotangent bundle to the Lie algebra $su(N)$. This means that we consider the space of pairs (P, Q) , where P and Q are $su(N)$ matrices (we identify Lie algebra $su(N)$ and the space of hermitian matrices by multiplying by $\sqrt{-1}$) with canonical symplectic structure

$$\Omega = tr(\delta P \wedge \delta Q). \quad (2.2)$$

Free motion is generated by quadratic hamiltonian

$$H_2 = \frac{1}{2} tr P^2 \quad (2.3)$$

On the $T^*su(N)$ acts unitary group $SU(N)$ by adjoint action on Q 's and coadjoint on P 's. This action preserves symplectic structure. If we identify the cotangent bundle with two copies of the Lie algebra with help of the Killing form we would get a moment map of this action in the form

$$\mu = [P, Q] \quad (2.4)$$

According to the standard prescription of hamiltonian reduction procedure we restrict ourselves to some level μ of momentum map and then factorise along the distribution of the kernels of the restriction of symplectic form. Generally these ones are spanned by the vectors, tangent to the orbits of the action of the stabilizer of point μ in the coadjoint representation. In our situation stabilizer depends on the number of different eigenvalues of matrix μ . In particular, if matrix μ has only two different eigenvalues, with multiplicities 1 and $N - 1$, then stabilizer will be $S(U(1) \times U(N - 1))$. We shall denote it by G_μ . Now we should resolve equation (2.4) modulo transformations from this group. It is easy to show, that if matrices P and Q satisfy condition $[P, Q] = \mu$ then we can diagonalize Q by conjugation by matrices from the G_μ . Let q_i be its eigenvalues. Then P turns out to be of the form (in the basis, where Q is diagonal and $\mu_{ij} = \nu(1 - \delta_{ij})$)

$$P = diag(p_1, \dots, p_N) + \nu || \frac{1 - \delta_{ij}}{(q_i - q_j)} || \quad (2.5)$$

where $\nu(N-1)$ and $-\nu$ are the eigenvalues of μ .

Symplectic structure on the reduced manifold turns out to be the standard one, i.e. p 's and q 's are canonically conjugate variables. Hamiltonian (2.3) is now hamiltonian for the natural system with potential V (2.1). Note that if we replace momentum matrix μ by the $F^+\mu F$ with F being diagonal unitary matrix, then reduced Hamiltonian H_2 wouldn't change. In fact, it is possible to obtain integrable system, starting from the Hamiltonian

$$H_k = \frac{1}{k} \text{tr} P^k \quad (2.6)$$

All H_k commute among themselves with respect to Poisson structure, defined by (2.2), thus yielding the complete integrability of the classical Calogero model. The restriction to the $su(N)$ case imply just the fixing of the center of mass of the system at the point zero.

2.2 Path integral for the Calogero model

Now we proceed to the quantization of the Calogero system. We consider path integral representation for the wave functions and by means of exact evaluation of path integral we obtain finite-dimensional integral formula for Calogero wavefunction (see [7]). We shall consider integral over the following set of fields. First, there will be the maps from the time interval $[0, T]$ to the direct product of three copies of $su(N)$ Lie algebra, actually to the product $T^*su(N) \times su(N)$. We denote corresponding matrix-valued functions as $P(t), Q(t)$ and $A(t)$ respectively. Then, we have fields $f(t)$ and $f^+(t)$ which are the CP^{N-1} - valued fields. Locally, these are N complex numbers f_i , which satisfy the condition $\sum_{i=1}^N f_i^+ f_i = 1$ and are considered up to the multiplication $f \rightarrow e^{i\theta} f$, $f^+ \rightarrow f^+ e^{-i\theta}$. We don't explain right now the supersymmetric version of this integral, so we just say that measure on P, Q fields comes from the symplectic structure (2.2), the A field measure comes from the Killing form on the Lie algebra, and the f, f^+ measure is again

the symplectic one, corresponding to the standard CP^{N-1} Fubini - Schtudi form $\Omega = p_1 i^* \nu \frac{1}{2\pi i} \delta f^+ \wedge \delta f$. Here, i^* means the restriction on the sphere $S^{2N-1} : f^+ f = N$, and p_1 is the factorization along the $U(1)$ action $f \rightarrow e^{i\theta} f$, $f^+ \rightarrow f^+ e^{-i\theta}$. Actually, as we will see, A is not a scalar field, it is Lie algebra-valued one-form (gauge field). More precise description of the measure will be given later, in the section 4.

Now let us define the action. We shall do it in two steps. At first we define the action for P, Q, A fields. It will be a sum of action of unreduced free system plus term which will fix the value of the moment map. Field A will play the role of the lagrangian multiplier.

$$S_{P,Q,A} = \int tr(iP\partial_t Q - \frac{1}{2}P^2 + iA([P, Q] + i\mu)) \quad (2.7)$$

Since μ has only two different eigenvalues with multiplicities 1 and $N-1$, it has the following form:

$$\mu = i\nu(Id - e \otimes e^+) \quad (2.8)$$

where ν is arbitrary (up to now) real number and we can choose vector e to be $e = \sum_{i=1}^N e_i$, e_i being the standard basis in C^N .

We shall evaluate the transition amplitude $\langle q' | \exp((t' - t)H) | q \rangle$. It is given by the path integral with boundary conditions. First we integrate out the A field. We observe that our path integral is invariant under the (small) gauge transformations

$$\begin{aligned} Q &\rightarrow gQg^{-1}, \quad P \rightarrow gPg^{-1} \\ A &\rightarrow A^g = gAg^{-1} + i(\partial_t g)g^{-1}; \\ g(t) &\in G_\mu, \quad g(0) = g(T) = Id \end{aligned} \quad (2.9)$$

We can enlarge the symmetry of our integral by introducing auxilliary fields f^+, f as follows. Let us rewrite term $tr(\mu(A^g))$ in terms of the vector

field $\bar{f} = g^{-1}e$, with e from the previous chapter. We get

$$S_f = \nu \int_0^T \bar{f}^+ (i\partial_t - A) \bar{f} \quad (2.10)$$

Obviously, S_f depends only on the class of \bar{f} 's modulo the total factor $e^{i\theta}$ if at the ends 0 and T θ 's coincide. Hence, it depends only on the CP^{N-1} valued field f , which corresponds to the \bar{f} . Therefore, our action $S = S_{P,Q,A} + S_{f,f^+}$ acquires the form

$$S = \text{tr}(iP\nabla_t Q - \frac{1}{2}P^2 + i\nu f^+ \nabla_t f), \quad (2.11)$$

where $\nabla_t Q = \partial_t Q + i[A, Q]$, $\nabla_t f = \partial_t f + iAf$. It is clear, that this action has $SU(N)$ gauge symmetry:

$$\begin{aligned} P &\rightarrow P^g = gPg^{-1}, \quad Q \rightarrow Q^g = gQg^{-1} \\ A &\rightarrow A^g = gAg^{-1} + i(\partial_t g)g^{-1} \\ f &\rightarrow gf, \quad f^+ \rightarrow f^+g^{-1}, \quad g \in SU(N) \end{aligned} \quad (2.12)$$

This invariance holds in the sense that amplitude is invariant under the $SU(N)$ action: $\langle Q, f | \dots | Q', f' \rangle = \langle gQg^{-1}, gf | \dots | g'Q'g'^{-1}, g'f' \rangle$.

As it usually goes in matrix models we diagonalize matrix $Q(t)$ by the conjugation by unitary matrix $g(t)$. We denote diagonalized matrix $Q(t)$ by \bar{Q} , and let q_i be its entries. Due to $SU(N)$ invariance of the action "angle" variables $g(t)$ decouple except the boundary terms. It is a gauge fixing so Faddeev-Popov determinant arises, which is nothing but the square of the Vandermonde determinant of q_i , $\Delta(\bar{Q})^2 = \prod_{i < j} (q_i - q_j)^2$.

We can keep it just by decomposing measure on the Lie algebra as the product of the Cartan measure on the q_i times $\Delta(\bar{Q})^2$ times the Haar measure on the unitary group and it should be done in each point of time interval, i.e. $\mathcal{D}Q = \prod_t d\bar{Q} \Delta(\bar{Q}_t)^2 dg$. Actually we get here an extra integration over Cartan subgroup. This implies that our gauge fixing is incomplete. We approve it by fixing the f field to be real and positive. This means that

among all the representatives ff of the point $f \in CP^{N-1}$ there should be one with real and positive components. This can be always achieved in the generic point due to the remaining action of the Cartan subgroup T^{N-1} . It is interesting to note that factor-space $\Delta_N = CP^{N-1}/T^{N-1}$ is nothing but the $(N-1)$ - dimensional simplex. It is parametrized by the variables $x_i = f_i^+ f_i$, $i = 1, \dots, N$, which satisfy conditions $x_i \geq 0, \sum_{i=1}^N x_i = N$. In fact, local coordinates on the CP^{N-1} are provided by x_i subjected by just described conditions and θ_i with conditions $-\pi < \theta_i < \pi, \sum_i \theta_i = 0$. These coordinates are well-defined when point x is inside of the internal region of Δ_N . The action S_f can be rewritten as

$$S_f = \int x_i \partial_t \theta_i - \sum_{i,j} \sqrt{x_i x_j} e^{i(\theta_i - \theta_j)} A_{ij}$$

Here, A_{ij} are matrix elements of A in the fundamental representation. In our approach we fix the value of x at the ends of time interval. This choice of polarisation on CP^{N-1} leads to the wavefunctions with support on some special points x , such that the monodromy of the ν 'th power of the Hopf line bundle along the $N-1$ -torus over x is trivial (Bohr-Sommerfeld condition). We will need the fact that after a quantization of CP^{N-1} with symplectic structure Ω some irreducible representation R_ν of $SU(N)$ appears. Its dimension

$$\dim R_\nu = \frac{(N + \nu - 1)!}{\nu!(N-1)!}$$

is the number of integer points in the corresponding simplex. This information permits us partially integrate out fields in our theory, namely, we integrate f^+, f and this make us left with the matrix element of the $\mathcal{P} \exp \int A$ in the representation R_ν . On the other hand, if we look at the integral over the non-diagonal part of the A field, we just recover constraint (2.4) and the Vandermonde - Faddeev - Popov determinant cancels. Here we don't introduce ghost fields, but in principle it can be done and the result will be the same. We have gauged Q to be in the Cartan subalgebra, therefore the term $[P, Q]$ has become orthogonal to the Cartan part of the A . This implies that

integration out this part yields

$$\prod_t \prod_{i=1}^{N-1} \delta(f_i^+ f_i - 1) = \prod_t \prod_{i=1}^{N-1} \delta(x_i - 1)$$

This produces after integrating out x_i fields some constraints on the boundary values of the fields x . The symplectic measure on the CP^{N-1} factorised along the T^{N-1} (i.e. the pushed forward measure to the Δ_N) is nothing, but $\prod_{i=1}^N dx_i \times \delta(\sum_{i=1}^N x_i - N)$. Geometric action $i\nu \int \delta^{-1} \Omega$ turns into $i\nu \int \sum_i \partial_t x_i = i\nu \sum_i x_i|_{t=T} - x_i|_{t=0}$. So the integral over CP^{N-1} valued part of the fields is done and the remaining integral over P, Q, A recovers Calogero amplitude for $g^2 = \nu(\nu - 1)$.

2.3 Lattice version of the Calogero model path integral

We wish to get the exact answer. To this end we present here a lattice version of this path integral, which will be calculable and in the reasonable continuum limit goes to the expressions written above. The main motivation for this model will become clear below, in the section concerning two-dimensional Yang-Mills theory. Let us point out, that this construction resembles Migdal's calculation in [8]. So, in order to calculate a transition amplitude

$$\langle Q', v' | \exp(-TH) | Q'', v'' \rangle, Q \in su(N), v \in R_\nu,$$

we consider a time-like lattice $0 < t_1 < \dots < t_K < T$. Let us denote its i 'th vertex as V_i and the link, passing from the V_i to V_{i+1} will be denoted as L_i . Actually we will calculate the answer for the Sutherland's model with potential

$$\mathcal{V} = \frac{(g/k)^2}{2 \sin^2 \frac{(x_i - x_j)}{k}},$$

Thus we have introduced an extra parameter k which will serve as an interpolating parameter. Pure Calogero model arises when $k \rightarrow \infty$. Our variables in hands will be the following: elements of the $SU(N)$ g_i, h_i , its irreducible

representations α_i and vectors $|v_i\rangle$ of some orthonormal basis in the representation R_ν . g_i and $|v_i\rangle$ are assigned at V_i , while α_i and h_i live on L_i . Lattice version of the amplitude is given by the following formula:

$$\begin{aligned} \langle g_0, v_0 | \dots | g_T, v_T \rangle = & \int \prod_{L_i, V_i} dg_i dh_i \sum_{\alpha_i} \sum_{v_i} \chi_{\alpha_i}(h_i g_{i+1} \times \\ & \times h_i^{-1} g_i^{-1}) \langle v_i | T_{R_\nu}(h_i) | v_{i+1} \rangle \times \\ & \times \dim(\alpha) \exp\left(-\frac{c_2(\alpha)}{k^2} \Delta t_i\right) \end{aligned} \quad (2.13)$$

It is supposed, that $g_i = \exp \frac{1}{k} X_i$, $X_i \in su(N)$, $c_2(\alpha)$ is a quadratic casimir in the irreps α , and integration over g, h goes with the Haar measure, normalised in such a way, that $\int dg = 1$. If we would replace $c_2(\alpha)$ by any invariant polynomial on the $su(N)$, we would get the answer for generalized Calogero model with Hamiltonian $H = \sum_k \epsilon_k H_k$. Now let us consider the continuum limit of the (2.13). First (2.13) is invariant with respect to the lattice gauge transformations:

$$\begin{aligned} g_i & \rightarrow U_i g_i U_i^{-1}, \quad h_i \rightarrow U_i h_i U_{i+1}^{-1}; \\ |v_i\rangle & \rightarrow T_{R_\nu}(U_i) |v_i\rangle \end{aligned} \quad (2.14)$$

We can partially fix this gauge freedom (gaining group-like Vandermondes), by diagonalizing g_i and imposing some conditions on $|v_i\rangle$. Integration over remaining Cartan part of gauge transformations projects vectors $|v\rangle$ onto the subspace of the fixed points of Cartan torus action on the R_ν . For this subspace to be non-trivial, ν should be divisible by N [9]. Let us consider contribution of one link. When $\Delta t_i \rightarrow 0$ the sum over representations turns into the $\delta(h_i g_{i+1} h_i^{-1} g_i^{-1} - Id)$. This implies that g_i and g_{i+1} have the same eigenvalues. Generically, we could fix the gauge, ordering this eigenvalues. So, in our gauge this yields $\delta(g_i - g_{i+1})$ and that h_i is in the Cartan subgroup, determined by g_i . In fact, we can set them to be equal to Id due to arguments from the previous paragraph. When $\Delta t_i \neq 0$ but still remains to be small, we expect that the main contribution to the integral (2.13) comes from g_i 's

which slowly change, when i varies, and from the h_i 's which are close to unity, i.e. $h_i = Id - i\Delta t_i A_i$, $iA_i \in su(N)$. Let us write $g_i = \exp(i\bar{Q}_i/k)$. Then $h_i g_{i+1} h_i^{-1} g_i^{-1}$ turns into

$$\exp i(\bar{Q}_{i+1} - \bar{Q}_i)/k - \frac{i\Delta t_i}{k} [A_i, e^{-i\frac{\bar{Q}_{i+1}}{k}}] e^{-i\frac{\bar{Q}_i}{k}}$$

Now we consider large k limit (so we restrict ourselves to the Calogero case. The treatment of the Sutherland system is simpler from the point of view of Yang-Mills theory). We will use formula, which will be proven below, in the chapter, concerning matrix models. Formula relates some limit of the characters and the Itzykson-Zuber integral, namely

$$\lim_{k \rightarrow \infty} \frac{\chi_{k \cdot \alpha}(\exp(i\frac{Q}{k}))}{\dim(k \cdot \alpha)} = \int_{SU(N)} dU \exp(\text{tr}(\bar{P} U Q U^{-1})) \quad (2.15)$$

where $k \cdot \alpha$ is the representation of $SU(N)$ whose signature is $k \cdot p_i$ and $\bar{P} = \text{diag}(p_1, \dots, p_N)$. In the limit $k \rightarrow \infty$ sum over representations becomes the integral over \bar{P} with the measure $d\bar{P} \Delta(\bar{P})^2$ (Vandermonde determinant comes from the $\dim(k \cdot)^2$. The integrand (contribution of V_i, L_i, V_{i+1}) is

$$\begin{aligned} & \int dU \exp(\text{tr}(U^{-1} \bar{P}_i U (\bar{Q}_{i+1} - \bar{Q}_i + \Delta t_i [A_i, \bar{Q}_{i+1}]) - \Delta t_i \bar{P}_i^2)) \times \\ & \times < v_i | T_{R_\nu}(\mathcal{P} \exp \int_{t_i}^{t_{i+1}} A) | v_{i+1} > \end{aligned}$$

It is clear that after redefinition $P = U^{-1} \bar{P}_i U - \frac{1}{N} \text{tr}(\bar{P}) Id$ and rewriting matrix elements of $\mathcal{P} \exp$ in the R_ν through CP^{N-1} path integral, we just get the path integral for the model (2.11) in the gauge Q is diagonal. After we have convince ourselves, that in the limit $\Delta t_i = t_{i+1} - t_i \rightarrow 0, k \rightarrow \infty$ this sum goes to the desired path integral with the action (2.11), we can integrate out field g_i . It turns out that integration over g_i , assigned to some vertex V_i , gives us after the redefinition $h_i \rightarrow h_{i-1} h_i$ the expression of the same type, as (2.13), so we can remove all internal vertices and we are left with the following integral representation for the amplitude:

$$\begin{aligned} & |\Delta(\exp(i\bar{Q}/k))| \sum_{\alpha} \dim(\alpha) \int dh \chi_{\alpha}(g_0 h g_T^{-1} h^{-1}) \\ & \times \exp(-\frac{T}{k^2} c_2(\alpha)) < v' | T_{R_\nu}(h) | v'' > \end{aligned} \quad (2.16)$$

This expression can be calculated even in the case of finite k and answer involves Clebsh-Gordon coefficients. It is obvious that if $|v' \rangle$ and $|v'' \rangle$ are not the fixed vectors of the Cartan torus action on R_ν then the amplitude vanishes. States are enumerated by the representations α (and by basis vectors in $Inv(\alpha \otimes \alpha^* \otimes R)$). The signature of the α divided by k gives the asymptotic momenta of the particles. One more observation concerning this simple example is that our wavefunction deals with a zonal spherical functions. In the zero coupling limits $\nu \rightarrow 0, 1$ wavefunctions turns into

$$\Psi_{\nu=0} = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \exp(i \sum_l \bar{P}_{\sigma(l)} \bar{Q}_l)$$

and

$$\Psi_{\nu=1} = \frac{\det || \exp(i \bar{P}_l \bar{Q}_m) ||}{\Delta(P)}$$

respectively (\mathcal{S}_N is a Weyl group for $SU(N)$) [7]. At generic coupling constants the system manifests the anyonic behaviour [10]

Let us mention that the integration over Q would lead to the Lax description of the integrable system. Lax equation implies that the Hamiltonian flow goes along the coadjoint orbit with the Hamiltonian A . It is clear that it is not the only way of the integration in this functional integral. For example, P field can be integrated out first. For the Hamiltonian H_2 we have a gaussian integral, and one immediately gets the particular example of the $c = 1$ gauged matrix model as an answer. Now the motion is not restricted on the one orbit and keeping in mind relation between the coadjoint orbits and representations of the group we can speak about the flow between the representations (actually, between all, not necessarily unitary ones). Thus the motion goes in the model space.

2.4 Calogero systems, related to generic root system

In this section we shall generalize this result to the case of generic root system. Calogero system, considered above, corresponds to the root system A_{N-1} . In

the next paragraph we remind the notion of the root system [11],[12] and, following [5], describe corresponding integrable systems.

Let (V, \langle, \rangle) be the vector space over R with inner product and let Δ be a finite set of vectors (roots) satisfying the following conditions:

$$\text{if } \alpha, \beta \in \Delta \text{ then } r_\alpha(\beta) := \beta - 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha \in \Delta; \quad (2.17)$$

$$\text{and } \frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \beta \rangle} \in \mathbf{Z} \quad (2.18)$$

Operators r_α entering into the definition are referred to as reflection in the root α , hyperplane, orthogonal to the root α is called the mirror, corresponding to the α . Root system Δ is referred to as simple root system if it cannot be decomposed into the direct sum of the two orthogonal root systems.

There is a classification of simple root systems, for example, the A_{N-1} root system is a set of vectors $e_{ij} = e_i - e_j, 1 \leq i, j \leq N$ in the $R^N \cap (\sum_i x_i = 0)$. With any semi-simple Lie algebra some root system is related (see [11]) and with any root system we can associate a dynamical system on V with the Hamiltonian

$$H_\Delta = \sum_{i=1}^N \frac{1}{2} p_i^2 + \sum_{\alpha \in \Delta} \frac{g_\alpha^2}{\langle q, \alpha \rangle^2} \quad (2.19)$$

Here $q = (q_1, \dots, q_N) \in V$, $g_\alpha^2 = \nu_\alpha(\nu_\alpha - 1)$ is a coupling constant and for the roots, belonging to the same orbits of the Weyl group, the value of the coupling constant should be the same. Actually, here also generalizations, including $\sin(\langle q, \alpha \rangle)$ or hyperbolic functions and/or extra quadratic potential are possible (in what follows we will call these generalizations as corresponding to the curved case).

Each such a system can be obtained as a result of hamiltonian reduction of the free system on the cotangent bundle to the corresponding Lie algebra. Extra Hamiltonians H_k (2.6) are replaced by the generators of the ring of invariant polynomials on the Lie algebra. Corresponding value of the

momentum map is chosen as follows:

$$\mu = \sum_{\alpha} \nu_{\alpha} (e_{\alpha} + e_{-\alpha}) \quad (2.20)$$

Here e_{α} are the roots of the corresponding Lie algebra.

Quantum calculation proceeds as before, the only difference is that instead of the group $SU(N)$ we have now generic semi-simple Lie group.

3 Two-dimensional Yang-Mills theory as a field theory limit of the Calogero system

In this section we will describe two things: first, we describe natural large N limit of the Calogero system, by that we mean replacing $SU(\infty)$ group by the central extension of the loop group $\mathcal{LSU}(N)$ - $SU(N)$ Kac-Moody (KM) algebra. We shall see that such a theory describes two-dimensional Yang-Mills theory with external source on a cylinder. Then we shall show, that compactification of the cylinder yields back N -particle Calogero system. The general procedure of deriving the field system starting from $SU(N)$ Calogero system should go as follows. At first one should consider the $SU(\infty)$ root system which is the two dimensional lattice on the complex plane (it corresponds to the representation of the $SU(\infty)$ as the $\mathcal{LSU}(\infty)$). We can conjecture that the corresponding Calogero system has Weierstrass \wp function as a potential. Then the $SU(N)$ KM root system embeds into lattice as the finite length strip of the roots. It is important that the KM spectral parameter which serve as a parameter on the loop appears already in the Baker-Akhiezer function for the $\wp(x)$ potential [13]. In what follows we shall omit these intermediate steps and start with the KM algebra from the very beginning, postponing the establishing of the relation between the \wp -function - type potential Calogero models and that one we shall obtain here.

3.1 Two-dimensional Yang-Mills theory via hamiltonian reduction

In the preceding chapters we have described construction of hamiltonian system, corresponding to some semi-simple Lie algebras g (in fact, only invariant non-degenerate inner product was needed). We considered "free" system on the cotangent bundle T^*g with Hamiltonian $H = V(P)$, where $P \in g^*$ (on the same footing we could consider Hamiltonian, which is a function on the g instead of g^*) and V is some invariant polynomial on g . Then we applied the procedure of hamiltonian reduction, corresponding to the adjoint - co-adjoint action of Lie group G on the T^*g . In order to quantize this reduced system we have introduced T^*g - valued fields and some auxilliary fields $A(t) \in g$ and $f(t), f^+(t) \in X_J$, with X_J denoting the co-adjoint orbit of the value $J \in g^*$ of moment map over which we wished to make the reduction. Here we generalize this construction to the infinite-dimensional case, namely we would like to substitute finite-dimensional Lie algebra by the Kac-Moody one.

We start from consideration of the central extension of current algebra $\mathcal{L}g$, where g is some semi-simple Lie algebra. The cotangent bundle to this algebra consists of the sets $(A, k; \phi, c)$, where ϕ denotes g -valued scalar field on the circle, c is a central element, A is g -valued one-form on the circle and k is the level, dual to c . The symplectic structure is the straightforward generalization of (2.2):

$$\Omega = \int tr(\delta A \wedge \delta \phi) + \delta k \wedge \delta c \quad (3.1)$$

Corresponding action of the loop group has the following form [14]: an element $(g(x), m | x \in S^1)$ acts as follows:

$$\phi \rightarrow g\phi g^{-1}, \quad c \rightarrow c + \frac{1}{2\pi} \int dx tr \phi g^{-1} \partial_x g \quad (3.2)$$

it was an adjoint part of the action, and the co-adjoint one is

$$A \rightarrow gAg^{-1} - \frac{k}{2\pi} \partial_x g \cdot g^{-1}, \quad k \rightarrow k \quad (3.3)$$

This action preserves the symplectic structure (3.1) and the corresponding moment map

$$(J, \bar{k}) = (\frac{k}{2\pi} \partial_x \phi + [A, \phi], 0) \quad (3.4)$$

has vanishing level. This implies that the infinite-dimensional analogue of X_J is highly degenerate one, and co-adjoint orbits of such a type have functional moduli space, namely the space of maps from the circle S^1 to the Cartan subalgebra of \mathfrak{g} modulo the global action of G Weyl group, (the symmetric group \mathcal{S}_N for $G = SU(N)$).

Now let us write down path integral for this system. To this end we need some extra fields: A_0 as a Lagrangian multiplier, and some fields which describe coordinates on the co-adjoint orbit of some element $J \in \mathfrak{g}^* \otimes \Omega^1(S^1)$. Let us denote this orbit as X_J , and corresponding field will be f . Let Ω_J be the symplectic structure on X_J and let μ_J be the moment map, corresponding to the co-adjoint action of $\mathcal{L}G$ on X_J . Then our action is

$$S = S_{\phi, A} + S_J$$

where

$$S_{\phi, A} = \int \int_{S^1 \times [0, T]} dx dt tr(i\phi \partial_t A_1 - \frac{\epsilon}{2} \phi^2 + A_0(\frac{k}{2\pi} \partial_x \phi + [A_1, \phi])) \quad (3.5)$$

and

$$S_J = \int \int_{S^1 \times [0, T]} (\delta^{-1} \Omega_J - dt A_0 \mu_J) \quad (3.6)$$

Action S is clearly invariant under the action of $\mathcal{L}G$, provided that Ω_J is integral, i.e. the orbit (X_J, Ω_J) is quantizable. Let us parametrize the orbit X_J by the group element $g(x, t)$. We perform a change of variables $A_1 \rightarrow \frac{1}{k} A_1$, it gives us the following action S :

$$S = \int_{\Sigma} tr(ik\phi F - d\mu \frac{\epsilon}{2} \phi^2 - J(x) g^{-1} (\partial_t + A_0) g) \quad (3.7)$$

Here Σ is some two-dimensional surface with distinguished measure $d\mu$ and the choice of the coordinates x, t can be considered as some choice of polarization. For example, it is possible to choose the holomorphic one, i.e.

substitute (x, t) by (z, \bar{z}) . In this case J has to be holomorphic g valued one-form. J can be always gauged to be the Cartan element (see above the description of degenerate orbits). Now we have to define the measure. To be consistent with finite-dimensional example we should define it in a way which distinguishes A_0 and A_1 . We prefer to follow [2] and define the measure for fields $A_i, i = 0, 1$ and ϕ as follows. The space of all G connections on the Σ is a symplectic one, with symplectic structure

$$\Omega = \frac{k}{4\pi^2} \int_{\Sigma} \text{tr}(\delta A \wedge \delta A) \quad (3.8)$$

It implies that on A 's there exists a natural symplectic measure. The measure on ϕ comes from the norm:

$$||\delta\phi||^2 = \int_{\Sigma} d\mu \frac{1}{2} \text{tr} \delta\phi^2, \quad (3.9)$$

which depends only on the measure $d\mu$ on the surface Σ . A question about the measure on $f(x, t)$ could be solved in the same way, i.e. we could choose a symplectic measure on the orbit. The problem is that the naive measure is just the $\prod_{x,t} dg_J(x, t)$ where dg_J is the (finite-dimensional) measure on the co-adjoint orbit $X_{J(x)}$ of the $J(x)$ in g^* . In general, dimension of $X_{J(x)}$ could be changed when one pass from one point x' to another x'' . Another problem is that possible quantum correction could destroy the invariance of the theory. So we prefer to choose here an appropriate ansatz for $J(x)$. Below we consider the case $G = SU(N)$, $g = su(N)$. First, we remind that Calogero-type systems correspond to the J with maximal proper stabilizer. Let us take

$$J(x)dx = \nu(x)dx(N \cdot Id - e(x) \otimes e^+(x)), \quad (3.10)$$

where $\nu(x)dx$ is some one-form on the circle and $e(x)$ is some field in the fundamental representation of $SU(N)$, which can be gauged as $e(x) = \sum_{i=1}^N e_i$, where e_i denotes a standard basis in C^N . The stabilizer of this element is the loop group $\mathcal{LS}(U(1) \times SU(N-1))$. This choice doesn't resolve the

problem of the measure. We restrict our consideration to $\nu = \sum_{i=1}^l \nu_i \delta(x-x_i)$. Now we have an orbit $X_J = \times_i (CP^{N-1}, \Omega_{\nu_i})$, where Ω_{ν_i} is the standard symplectic form on CP^{N-1} ω_F , multiplied by ν_i . Let \mathcal{R}_{ν_i} be the representation, which corresponds ' to $(CP^{N-1}, \Omega_{\nu_i})$ after quantization by Kirrilov construction. Now we can take measure on X_J to be $\prod_i \Omega_{\nu_i}^{\wedge N-1}$. This will give us a desired path integral measure. States in this theory will contain vectors $\otimes_i |v_i\rangle$ in the tensor products of representations \mathcal{R}_{ν_i} . In fact, states are functionals $\Psi(A)$ of the gauge field A_1 on the circle, multiplied by the vectors $|v_i\rangle$, invariant under the action of gauge group:

$$\begin{aligned} \Psi(A) \otimes \otimes_i |v_i\rangle &\rightarrow \\ \rightarrow \Psi(A^g) \otimes \otimes_i T_{\mathcal{R}_{\nu_i}}(g(x_i)) |v_i\rangle \end{aligned} \quad (3.11)$$

Here $T_{\mathcal{R}_{\nu_i}}(g)$ is an image of element g in the representation \mathcal{R}_{ν_i} . It is obvious, that the only gauge invariant of the gauge field A_1 in the interval (x_i, x_{i+1}) is the monodromy

$$h_{i,i+1} = \mathcal{P} \exp \int_{x_i}^{x_{i+1}} A_1$$

It is invariant of the gauge transformations, which are trivial at the ends of interval (x_i, x_{i+1}) . So, wavefunctions are functions of $h_{i,i+1}$ and we must take invariants of the $SU(N)$ action

$$\begin{aligned} \Psi(h_{i,i+1}) \otimes \otimes_i |v_i\rangle &\rightarrow \\ \rightarrow \Psi(g_i h_{i,i+1} g_{i+1}^{-1}) \otimes \otimes_i T_{\mathcal{R}_{\nu_i}}(g_i) |v_i\rangle \end{aligned}$$

For example, if we have only one representation R , i.e. $l = 1$, then the Hilbert space is just

$$\oplus_{\alpha} \text{Inv}(\alpha \otimes \alpha^* \otimes R)$$

where sum goes over all irreps of $SU(N)$. When R is trivial, this gives us just the space of all characters.

Now we proceed to the calculation of transition amplitudes. It is easy to show that it is given by the path integral over the fields ϕ, A on the cylinder

$S^1 \times (0, T)$ with fixed boundary monodromies $h_{i,i+1}$ and vectors $|v_i\rangle$:

$$\begin{aligned}
& \langle \{h'_{i,i+1}, v'_i\} | \exp(-TH) | \{h''_{i,i+1}, v''_i\} \rangle = \\
& \int \mathcal{D}A \mathcal{D}\phi \exp \int \text{tr} (ik\phi F - \frac{\epsilon}{2}\phi^2) \times \\
& \times \prod_i \langle v'_i | \mathcal{P} \exp \int_{C_i} A | v''_i \rangle_{\mathcal{R}_{\nu_i}}, \tag{3.12}
\end{aligned}$$

here C_i is a straight contour on the cylinder, which goes along the time axis from the point $(x_i, 0)$ to the point (x_i, T) .

This path integral can be easily reduced to the finite-dimensional one, using techniques, developed in [2], [8], [15]. This reduction appears due to the possibility of cutting the cylinder along the contours C_i . Then, path integral on the disk $D_{i,i+1}$ with boundary $\partial D_{i,i+1} = C_{i+1} - ([x_i, x_{i+1}], T) - C_i + ([x_i, x_{i+1}], 0)$, (signs + and - mean orientation) is given by the following sum

$$\begin{aligned}
I_{D_i} &= \sum_{\alpha} \chi_{\alpha}(h'_{i,i+1} g_{i+1} h''_{i,i+1}{}^{-1} g_i^{-1}) \\
&\times e^{-\frac{TL_{i,i+1}}{k^2} c_2(\alpha)} \tag{3.13}
\end{aligned}$$

where $L_{i,i+1} = x_{i+1} - x_i$, $c_2(\alpha)$ is the quadratic casimir in the irreps α , g_i is the monodromy along the C_i and over g_i we must integrate with the Haar measure. So the answer has the following form:

$$\int \prod_i dg_i \langle v'_i | T_{\mathcal{R}_{\nu_i}}(g_i) | v''_i \rangle I_{D_i} \tag{3.14}$$

In principle, this integral over product of group manifold can be done and the final answer includes the Clebsh-Gordan coefficients. The general formula is rather complicated. The partition function calculation will involve integrals of the products of characters of representations \mathcal{R}_{ν_i} and α , hence the answer will include multiplicities of the enterings of representations \mathcal{R}_{ν_i} 's into the tensor products $\alpha \otimes \alpha^*$. Let us denote them as $D(\mathcal{R}_{\nu_i}, \alpha)$. We observe that the spectrum of this theory depends on the \mathbf{Q} - relations between the $L_{i,i+1}$'s

and $D(\mathcal{R}_{\nu_i}, \alpha)$. If the latters are \mathbf{Q} - independent, then for generic R the spectrum is given by

$$E_{\alpha_1, \dots, \alpha_l} = \sum_i \frac{L_{i,i+1}}{k^2} c_2(\alpha_i)$$

If lenghts $L_{i,i+1}$ are linearly independent over \mathbf{Q} , then the spectrum is quasi-continuous (for this effect to take place l have to be greater then one).

In the case $l = 1$, the case of the only one representation R , we have an integral of the type

$$I(h', h'')_{\alpha, R} = \int dg \chi_\alpha(h' g h'' g^{-1}) \langle v' | T_R(g) | v'' \rangle$$

It is given by the sum over the orthonormal basis θ in the space of all invariants $Inv(\alpha \otimes \alpha^* \otimes R)$:

$$\begin{aligned} & \sum_{\theta} C_{\langle i_\alpha | \otimes \langle j_{\alpha^*} | \otimes \langle v'_R |}^{\theta} \langle j_\alpha | h' | i_\alpha \rangle \times \\ & \times C_{\langle k_{\alpha^*} | \otimes \langle m_\alpha | \otimes \langle v''_{R^*} |}^{\theta} \langle k_\alpha | h'' | m_\alpha \rangle, \end{aligned} \quad (3.15)$$

where C_{\dots} are in fact the Clebsh-Gordon coefficients and $|i_\alpha \rangle, \dots, |m_\alpha \rangle$ run over basis in the irreps α , while $|i_{\alpha^*} \rangle$ etc. run over the dual basis.

It explains the correspondence between the Calogero-Moser model, the Sutherland model and Yang-Mills theory with inserted Wilson line. To get the Calogero answer we must take h', h'' to be diagonal matrices

$$h = \text{diag}(\exp(i \frac{q_i}{k}))$$

and take large k limit, keeping $\bar{P}_i = \frac{\bar{p}_i}{k}$ finite, where \bar{p}_i 's are numbers, connected with the lenghts of the columns of the Young tableau of α (see below). Then we will recover wavefunction of the Calogero model, corresponding to the state with asymptotic momenta \bar{P}_i .

Let us justify once again this correspondence by resolving the constraint, coming from the fixing the value of the moment map at the point $J(x) =$

$\nu\delta(x)(Id - e \otimes e^+) \ (l = 1)$. We can gauge field A_1 to the constant diagonal matrix, and from constraint after simple calculations we conclude, that diagonal part of ϕ is some constant map from the circle to the Cartan subalgebra - $\phi_{ii} = p_i$, while non-diagonal part is given by the following formula:

$$\phi_{ij} = \nu \left[sgn(x) - cot\left(\frac{\pi(q_i - q_j)}{k}\right) \right] e^{-\frac{2\pi ix}{k}(q_i - q_j)} \quad (3.16)$$

where q_i are eigenvalues of A_1 , $0 < x \leq 1$ is a coordinate on the circle and $sgn((x))$ is a multivalued step-function. This expression is periodic, though it jumps when x pass through some point $x = 0$ on the circle. This gives a Hamiltonian

$$H_2 = \sum_i \frac{p_i^2}{2} + \sum_{i \neq j} \frac{(\nu/k)^2}{2 \sin^2(\frac{q_i - q_j}{k})} \quad (3.17)$$

In fact, more rigorous proof involves the consideration of two Wilson lines in the same representation. Then constraint can be resolved in the class of continuous ϕ 's. In the limit when the distance between lines goes to zero we obtain just 3.17. This establishes the desired relation once again. (As usual, reduction at classical level doesn't impose any conditions on the ν and we don't see the quantum shift $\nu^2 \rightarrow \nu(\nu - 1)$.)

3.2 Relation with matrix models

Let us investigate in more details the dependence on the k parameter. In what follows we set all sources to be zero, so we take pure YM system. We see that if we make a change of variables

$$A_1 \rightarrow \frac{1}{k} A_1, \ \psi_1 \rightarrow \frac{1}{k} \psi_1$$

then path integral acquires the following form:

$$k^{-N} \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\phi \exp\left(\int_{\Sigma} tr\left(k\left(\frac{1}{2}\psi \wedge \psi + i\phi F\right)\right) + \frac{\epsilon}{2}\phi^2\right) \quad (3.18)$$

Here ψ is a fermionic field, which is a superpartner of A , i.e. odd-valued one-form in adjoint representation. It serves here just for the proper definition of the measure $\mathcal{D}A$. After simple integrating out ϕ field integral turns into two-dimensional QCD path integral:

$$k^{-N} \int \mathcal{D}A \exp\left(\int_{\Sigma} \text{tr}\left(-\frac{k^2}{2\epsilon} F_{\mu\nu}^2\right)\right) \quad (3.19)$$

The thing which deserves some comment is the appearance of the factor k^{-N} (it is on the genus one surface, in case of genus g the factor will be $k^{-(N^2-1)(g-1)}$). Actually it comes from the simple observation that if one considers an integral over symplectic manifold \mathcal{M} of the form $\exp(\omega)$ then $\int \exp(k\omega) = k^{\dim \mathcal{M}/2} \int \exp(\omega)$. In our situation integral partially localizes on the moduli space of flat connections in the principal $SU(N)$ -bundle over Σ . We might say that we have normalized our path integral in such a way that its perturbative in ϵ part, which is responsible only for the integral over the moduli space remains finite then $k \rightarrow \infty$. It is easy to show that if we take our space-time Σ to be a torus of area $\int_{\Sigma} d\mu = T$ then in the absence of the source J the partition function is given by the simple generalization of the Migdal-Witten formula [2],[8],[17] :

$$Z = k^{-N} \sum_{\alpha} \exp\left(-\frac{\epsilon T}{k^2} c_2(\alpha)\right) \quad (3.20)$$

where sum goes over all irreducible representations of $SU(N)$ and $c_2(\alpha)$ is a quadratic casimir in the representation α . The wave function in the q representation is now a function of the monodromy $U = g_A(2\pi)$, invariant under conjugation, so it is a character and traces in the irreducible representations provide basis in the Hilbert space. Each such a trace is a common eigenfunction for all invariant polynomials on the $su(N)$ Lie algebra considered as an operators in the theory. This statement is general and is valid in any two-dimensional gauge theory. We can consider slightly generalized

Yang-Mills system (GYM), which corresponds to the inclusion of the higher hamiltonians (2.6). This system has a form

$$\mathcal{L} = \int_{\Sigma} tr(k\phi F + \sum_{i=2}^N \epsilon_i \frac{\phi^i}{i}) \quad (3.21)$$

Let us consider the solution of the corresponding Schrödinger equation with initial condition $\Psi(U)|_{t=0} = \delta(U - Id)$, $U \in SU(N)$. It is easy to show, using [2], that

$$\Psi(U)|_{t=a} = \sum_{\alpha} \exp(-a \sum_i \frac{\epsilon_i}{ik^i} c_i(\alpha)) dim(\alpha) \chi_{\alpha}(U). \quad (3.22)$$

Now we would like to consider large k limit. For this we shall make a small digression and remind some simple facts about the irreps of the $SU(N)$ ([18]). Each finite-dimensional irreducible representation α of the $SU(N)$ group is characterised by the set of integers $(p_1 \geq \dots, \geq p_N \geq 0)$ - the signature of the representation. Let us introduce conventional variables $\bar{p}_i = p_i + N - i$. Then dimension, character and casimirs of the representation α are equal to

$$\begin{aligned} dim(\alpha) &= \frac{\Delta(\bar{p}_i)}{\Delta(i)}; \\ \chi_{\alpha}(U) &= \frac{det||\beta_j^{\bar{p}_i}||}{\Delta(\beta_j)}, \text{ where } U = g \text{diag}(\beta_1, \dots, \beta_N)g^{-1}, g \in SU(N); \\ c_l(\alpha) &= \sum_{i=1}^N \bar{p}_i^l \prod_{j \neq i} (1 - \frac{1}{\bar{p}_i - \bar{p}_j}) \end{aligned} \quad (3.23)$$

Generic connection on the circle can be gauged to the diagonal constant matrix $A = \text{diag}(iQ_1, \dots, iQ_N)$, therefore the value of the wavefunction at the generic point $U = \text{diag}(\exp(\frac{2\pi i}{k}Q_j))$ is

$$\Psi(Q_j, a) = \frac{k^{\frac{N(N-1)}{2}}}{\Delta(Q_i)} (1 + \mathcal{O}(\frac{1}{k})) \sum_{\bar{p}_i} \exp(-a \sum_l \epsilon_i \frac{P_l^i}{i}) \frac{det||\exp(iP_i Q_j)||}{\Delta(P_j)} \quad (3.24)$$

where $P_i = \frac{\bar{P}_i}{k}$. In the limit $k \rightarrow \infty$ this sum turns into the integral over the \bar{P} , which is nothing (modulo some trivial factors, coming from the slightly different integration region), but the partition function for the Generalized Kontsevich Model (GKM) (see [19],[20]). Here we have used Itzykson-Zuber integration formula

$$\int_{SU(N)} dU \exp(\text{tr}(U \bar{P} U^{-1} \bar{Q})) = \frac{\det ||e^{i P_i Q_j}||}{\Delta(\bar{P}) \Delta(\bar{Q})}$$

It would be interesting to realize, what kind of integrable system, what kind of τ - function arises if Wilson loops are included. Let us mention (it follows easily from the [2]) that perturbative in ϵ_i part of the partition function for GYM is a generating function for intersection pairings of cohomology classes of the moduli space of flat $SU(N)$ - connections, namely

$$Z_{\text{pert}} = k^{-N} \int_{\mathcal{M}} \exp(k\omega + \sum_i \epsilon_i \Theta_i) \quad (3.25)$$

where ω is the symplectic structure of \mathcal{M} and Θ_i are characteristic classes of the tautological bundle $\mathcal{A}_0 \rightarrow \mathcal{M}$, with \mathcal{A}_0 being the space of all flat $SU(N)$ -connections on the Σ .

It is interesting to note that $k \rightarrow \infty$ limit yields the quasi-continous spectrum of $\frac{c_l(\alpha)}{lk^l}$ and gives back the spectrum in the Calogero model, corresponding to the $\text{tr}(\frac{1}{l} \bar{P}^l)$. Additionally, we see the interpolation between the wave function in the Calogero model with zero coupling $\Psi_{\bar{P}}$ and the wave function in the 2D YM theory χ_α where α is a representation, which appears after quantization of the $SU(N)$ co-adjoint orbit of $\frac{\bar{P}}{k}$.

Let us again consider two dimensional Yang-Mills theory on the cylinder $\Sigma = S^1 \times (0, T)$. We normalize measure $d\mu$ in such a way that the whole area of Σ is just $2\pi T$. We have seen in the previous subsection that in the limit $k \rightarrow \infty$ wavefunctions of the YM system go into the quantum mechanical wavefunctions, corresponding to the system of N free particles on the real line, so we have got a kind of compactification in 2d YM theory.

In principle, it is not a trivial question whether it is possible to define the procedure of compactification for topological theory, like topological Yang-Mills theory or even for the physical two dimensional gauge theory which is invariant under the action of the area preserving diffeomorphisms. In that kind of theory, small radius of the spatial direction corresponds to the small area of the space-time manifold - the only (except genus and number of holes) invariant of two dimensional symplectomorphisms group. In 2d YM theory small area limit corresponds to the contribution of the moduli space of flat connections to the partition function.

Let us look at the duality picture between the motion along and across the orbits in the KM case. It is easily seen that integrating over A_1 we get the motion along the ϕ (adjoint!) orbit. On the other hand if we integrate over ϕ from the very beginning then usual YM-like (and higher order) terms in the curvature appear. $k \neq 0$ coadjoint KM orbits are labelled by the conjugacy classes of the finite dimensional group, so if it happens that there is a flow of these classes then we have got the motion, transversal to the orbits. Here it is just the case because we have nontrivial in general evolution of the eigenvalues of the monodromy matrix.

4 Supersymmetry

It is well known that the safest way to deal with path integral measure is to rewrite it using superpartners of matter fields. For example, 2d YM partition function is rewritten as [2]:

$$Z = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\phi \exp\left(-\int \text{tr}\left(\frac{1}{2}\psi \wedge \psi + i\phi F + \frac{\epsilon}{2}\phi^2\right)\right) \quad (4.1)$$

In the case of the YM system with the source J which is not conserved in a sense of the $SU(N)$ gauge theory, the procedure must be more elaborated. In what follows we consider finite-dimensional case, i.e. the Calogero system.

The natural supersymmetrisation of quantum mechanical system involves introducing of the superpartners of the variables P, Q in such a way that zero modes of them represent one-forms on the symplectic manifold. Keeping in mind the equivariant $SU(N)$ derivative, acting on the equivariant forms on the $T^*su(N) \times CP^{N-1}$ and extending it to the loop space, we get the following action of the supersymmetry algebra:

$$\begin{aligned}
\mathcal{Q}P &= \psi_P, \quad \mathcal{Q}Q = \psi_Q, \quad \mathcal{Q}A = 0 \\
\mathcal{Q}\psi_P &= i\nabla_t P, \quad \mathcal{Q}\psi_Q = i\nabla_t Q - P, \\
\mathcal{Q}f &= \Psi_f, \quad \mathcal{Q}f^+ = \Psi_f^+ \\
\mathcal{Q}\Psi_f &= i\partial_t f, \quad \mathcal{Q}\Psi_f^+ = i\nabla_t f^+
\end{aligned} \tag{4.2}$$

Path integral measure

$$\mathcal{D}\mu = \mathcal{D}P\mathcal{D}Q\mathcal{D}A\mathcal{D}\psi_P\mathcal{D}\psi_Q\mathcal{D}f^+\mathcal{D}f\mathcal{D}\Psi_f^+\mathcal{D}\Psi_f$$

is well-defined due to supersymmetry.

Action, which includes fermions, can be written (locally) as

$$S = \{\mathcal{Q}, V\},$$

with

$$\begin{aligned}
V &= \frac{1}{2} \int (tr(P\psi_Q - Q\psi_P) + \\
&+ \Theta_i \Psi_{f_i} - \Theta_i^+ \Psi_{f_i}^+)
\end{aligned} \tag{4.3}$$

It gives

$$\begin{aligned}
S &= \int (tr(\psi_P\psi_Q) + \Omega_{\bar{i}j} \Psi_{f_i}^+ \Psi_{f_j} \\
&+ i tr(P\nabla_t Q - \frac{1}{2}P^2) + if^+ \nabla_t f)
\end{aligned} \tag{4.4}$$

$$\tag{4.5}$$

where Ω_{ij} is a symplectic form on the CP^{N-1} , i.e. Fubini-Shtudi form, multiplied by ν , and Θ is a (locally defined) symplectic potential, i.e.

$$\Omega_{ij} = \partial_i \Theta_j - \partial_j \Theta_i$$

. In general, our operator \mathcal{Q} is not nilpotent, but it is nilpotent when restricted onto the space of observables, invariant under the action of some group, which is defined below. This proves the equivariant closeness of the action and permits to use the localization techniques to evaluate path integral (at least, partition function).

Due to general arguments, we can add to action any term $t\{\mathcal{Q}, \mathcal{V}'\}$ and the answer will be independent of t , provided that we calculate expectations of \mathcal{Q} closed objects. This gives the way of exact evaluation of path integral [16]. The equivariant derivative which we have written corresponds to the action of the semidirect product of \mathcal{R} and $\mathcal{LSU}(N)$ groups on the loop space $\mathcal{LT}^*su(N) \times CP^{N-1}$. An element $(g(t), \tau), g(t) \in \mathcal{LSU}(N), \tau \in \mathcal{R}$ acts as follows:

$$Q(t) \rightarrow g(t)(Q(t + \tau) - \tau P(t + \tau))g(t)^{-1}, P(t) \rightarrow g(t)P(t + \tau)g(t)^{-1}$$

$$f(t) \rightarrow g(t)f(t + \tau), f^+(t) \rightarrow f^+(t + \tau)g(t)^{-1}$$

Loop space (pre)symplectic forms $\int tr(\psi_P \psi_Q)$ and $\int \Omega_{ij} \Psi_{f_i}^+ \Psi_{f_j}$ are invariant under this action. They can be equivariantly extended to become equivariantly closed forms. This extension is nothing, but our action.

It is quite surprising from the finite-dimensional point of view that in the case of the reduction under the zero level of the moment map there is another realisation of the supersymmetry algebra, which corresponds to the dimensional reduction of the YM theory. We know that in this situation we don't need extra degrees of freedom, corresponding to the co-adjoint orbit of the moment value, since it vanishes. So we are left with the fields P, Q and A . The action reads as follows:

$$Z = \int \mathcal{D}P \mathcal{D}Q \mathcal{D}A \mathcal{D}\psi_Q \mathcal{D}\psi_A \exp\left(\int tr(iP\partial_t Q + \psi_Q \psi_A + A[P, Q] - \frac{\epsilon}{2}P^2)\right) \quad (4.6)$$

ϵ is arbitrary constant.

Second supersymmetry algebra is realized as follows:

$$\begin{aligned} \mathcal{Q}A &= \psi_A, \mathcal{Q}Q = \psi_Q, \mathcal{Q}P = 0 \\ \mathcal{Q}\psi_A &= i\nabla_t P, \mathcal{Q}\psi_Q = [Q, P], \end{aligned} \quad (4.7)$$

5 Relation with the rational KP and KdV solutions

It is known for a long time [13] that the rational, trigonometric and elliptic solutions of KdV and KP are closely related with the Calogero system, for example the rational KdV solution is

$$u(x, t) = \sum_i \frac{g_i^2}{(x - x_i(t))^2} \quad (5.1)$$

where the motion of the poles goes according to the Calogero Hamiltonians [5].

More generally if one has some elliptic spectral curve \mathcal{G} with the periods related to the periods of the corresponding Weierstrass function and maps it into the covering algebraic curve \mathcal{G}_n defined by the characteristic equation

$$\det(\lambda + L(x_i(t), \alpha)) = 0 \quad (5.2)$$

where L is Lax operator, (the matrix P in our notations),

$$L_{ij} = p_i \delta_{ij} + 2(1 - \delta_{ij})F(x_i - x_j, \alpha) \quad (5.3)$$

$$F(x, \alpha) = \frac{\sigma(x - \alpha)e^{\xi(\alpha)x}}{\sigma(\alpha)\sigma(x)} \quad (5.4)$$

$$\xi(z) = \frac{\sigma'(z)}{\sigma(z)} \quad (5.5)$$

α is the point on the spectral curve, σ is the Weierstrass function, then this curve can be mapped into n -dimensional torus whose coordinates are nothing but the angle variables for the n -particle system with pairwise interaction via the Weierstrass potential. The zeroes of the elliptic KdV tau function which is the θ function of the curve G_n now define the coordinates of the particles

$$\prod_i \sigma(x - x_i(t)) = 0 \quad (5.6)$$

The same is true for the KP elliptic solutions. The number of the particles equals to the genus of the curve \mathcal{G}_n . The degeneration of the spectral curve when one or both periods tends to infinity gives rise to the trigonometric or rational interaction potentials. The transition from the trigonometric to rational case can be interpreted as the transition from the group to the algebra [21]. We don't have at a moment the transparent picture for the transition from the elliptic to trigonometric case. From our consideration it follows that k dependence could, in principle, play the role of the interpolating parameter between \wp , inverse sine squared and inverse square potential, or, between Lie algebra and Lie group, since the relevant group elements in hands look like $diag(e^{\frac{2\pi i}{k}Q_i})$, which is a group-like element for finite k and seems to be Lie algebraic one in the limit $k \rightarrow \infty$

Another point which needs further clarification is the connection of the KdV or KP structures with the two-dimensional YM. In fact the evolution of the rational KdV solution is the motion along the Virasoro coadjoint orbit which is the dual object to the space of the complex structures of Riemann surfaces and the corresponding GKM models feel the structure of this moduli space. From the other hand two dimensional YM feels the structure of the

moduli space of the flat connections which was mentioned above. Therefore having the relation between Calogero models and rational or in general elliptic solution of KdV from the one hand and with YM from other hand we should find the relation between these two objects. It should be formulated in terms of the relation between the moduli space of the flat connections and the moduli space of Riemann surfaces. The motion corresponding to the rational solutions have to be reformulated as some motion in the phase space of YM.

6 Conclusions

Let us summarize the picture just considered. We have shown that the proper generalization of the Calogero-Moser system leads to the two dimensional YM theory with a source. There is a relation between the wave functions of Calogero and compactified YM. So having this relation at hands it is natural to ask all the standard YM questions in terms of Calogero-Moser and vice versa. Another point is the path integral definition for the (at least finite dimensional) integrable systems. Different choices of the polarization of the phase space, i.e. different integration orders in the functional integral, manifest the duality structure of the systems at hand - we saw directly two different flows along and across the orbits.

Many important questions remained beyond the scope of the paper. It is important to realize the proper meaning of the generalization of Calogero-Moser to the curved case, the corresponding picture for the Virasoro algebra, introduction of the oscillator potential which has the meaning of the mass term in YM and so on. We will consider it in further publications. The problems related with the anyonic interpretation and the relation with the collective field theory [22] have not been discussed.

A point which also to be mentioned is that in the infinite coupling con-

stant limit in the Calogero system with the oscillator term the particles are localized at the equilibrium points and thus we have no nontrivial dynamics in the space of the observables. It is known that the equilibrium position for the system above and the system of the Coulomb particles in the oscillator potential coincide so in the infinite coupling constant limit the systems are closely related. From the other hand the Coulomb particles represent the equivalent picture for correlators in two dimensional CFT so the Calogero particles imitate the positions of the vertex operators. Thus the infinite coupling constant limit corresponds the case of the fixed positions of vertex operators on the world sheet or in other words we have no moving branching points.

When the paper was completed, we have become noticed about the works [23],[24] on related subjects.

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